Bounds on Momentum Dependence of Phase Shifts and Magnitude of Coupling Constants

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The analytic structure of partial-wave amplitudes is used to derive a dispersion relation for the scattering phase shift. This dispersion relation is used to obtain a lower bound on the momentum derivative of the phase shift. The bound depends on an integral over the unphysical (left-hand) cut in the momentum squared plane and can be expressed in terms of the number of zeros of the real part of the S matrix along the unphysical cut. Stronger bounds are also presented involving the position of these zeros and the locations and widths of resonances and virtual states. The same approach is used to obtain limits on the magnitude of coupling constants.

1. INTRODUCTION

IT has been shown by Wigner[1] that the momentum derivative of the scattering phase shift must exceed a certain limit if the potential vanishes beyond a certain distance. A similar limit has been obtained by Goebel, Karplus, and Ruderman[2] for a relativistic neutral two-particle system, again under the restriction that the interaction be of finite range.

Since, fundamentally, Wigner's theorem is a consequence of causality, it should be possible to prove it within the framework of S-matrix theory, without introducing explicitly an interaction range but by making use instead of the analytic structure of the S matrix. This is the aim of the present paper wherein the analytic structure of the S matrix is determined from the assumption that the partial-wave amplitudes have no singularities other than those that follow from the Mandelstam representation. As a by-product we also obtain upper limits for the coupling constants, similar to those obtained by Ruderman.[3]

The bounds on the coupling constant or the momentum derivative of the phase shift involve, in their weakest form, the number of zeros of the real part or the imaginary part of the S matrix on the unphysical cut. Stronger bounds are also presented involving the positions of these zeros, and/or the positions of the zeros of the S matrix itself in the entire complex plane (i.e., the locations and widths of resonances and virtual states), and/or the imaginary part of the phase shift along the inelastic cut.

2. DISPERSION RELATION FOR PHASE SHIFT

Consider the functions $\delta(z)$, $S(z)$, $f(z)$, and $\rho(z)$ of the complex variable z related by[4]

$$e^{2i\delta(z)}=S(z)=1+2i\rho(z)f(z). \tag{1}$$

The phase shift $\delta(x)$, the S matrix $S(x)$, and the partial-wave amplitude $f(x)$ are the boundary values of the corresponding functions of z as z approaches x from above, where x is real and positive and equal to the square of the three-momentum of the scattering particles in their barycentric frame. $\rho(z)$ is a kinematic factor. We consider the scattering of pseudoscalar isovector particles (pions) as an example. Then

$$\rho(z)=z^{1/2}(z+1)^{-1/2}, \tag{2}$$

where we have set the pion mass equal to unity. Here and in the following we define the roots $z^{1/2}$ and $(z+1)^{1/2}$ such that their imaginary parts are non-negative in the cut plane with branch lines between 0 and $+\infty$ and between $-\infty$ and -1. We note for future reference that $S(z)$ and $f(z)$ are real functions

$$S^*(z^*)=S(z),\quad f^*(z^*)=f(z), \tag{3}$$

whereas

$$\rho^*(z^*)=-\rho(z),\quad \delta^*(z^*)=-\delta(z). \tag{4}$$

It follows from the Mandelstam representation that $f(z)$ is analytic in the z plane cut as above. Hence by Eq. (1), $S(z)$ is analytic in the cut plane, and $\delta(z)$ is analytic in the cut plane except for logarithmic branch points at the zeros of $S(z)$. Let the zeros of the S matrix on the negative real axis lie at $-a_p$, $a_p>0$, those on the positive real axis lie at b_r, $b_r>0$, and those in the complex plane lie at α_s and α_s^* [it follows from Eq. (3) that the complex zeros appear in pairs]. The a_p for different p, b_r for different r, and α_s for different s are not assumed to be necessarily distinct. Then the function $\bar{S}(z)$, defined by

$$\bar{S}(z)=S(z)/D(z),$$
$$D(z)=\textstyle\prod_p(z+a_p)\prod_r(z-b_r)\prod_s(z-\alpha_s)(z-\alpha_s^*), \tag{5}$$

is analytic in the cut plane and has no zeros.

Consider therefore the function

$$\eta(z)=\frac{1}{2i\sqrt{z}}\ln\bar{S}(z)=\frac{\delta(z)}{\sqrt{z}}-\frac{1}{2i\sqrt{z}}\ln D(z), \tag{6}$$

where we take that branch of $\ln\bar{S}(z)$ that is real in the gap $-1<z<0$. Since $\bar{S}(z)$ has no zeros, $\eta(z)$ is analytic

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[1] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
[2] C. J. Goebel, R. Karplus, and M. A. Ruderman, Phys. Rev. **100**, 240 (1955).
[3] M. A. Ruderman, Phys. Rev. **127**, 312 (1962).
[4] We omit throughout the subscripts on δ, S, and f, that identify their angular momentum and isotopic spin.

in the cut plane; furthermore, $\eta(z)$ vanishes at infinity provided that $\delta(z)$ behaves at infinity no worse than $z^{1/2-\epsilon}$, $\epsilon>0$. We may therefore apply to $\eta(z)$ Cauchy's theorem and obtain

$$\eta(z)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{[\eta(x)]}{x-z}dx+\frac{1}{\pi}\int_{0}^{\infty}\frac{[\eta(x)]}{x-z}dx, \quad (7)$$

where

$$[\eta(x)]=\{\eta(x+i0)-\eta(x-i0)\}/(2i). \quad (8)$$

It follows from Eq. (6) that for $x>0$

$$[\eta(x)]=-\frac{1}{4\sqrt{x}}\ln\{\bar{S}(x)\bar{S}^*(x)\} =\frac{\delta_I(x)}{\sqrt{x}}+\frac{1}{2\sqrt{x}}\ln|D(x)| \quad (9)$$

and for $x<0$

$$[\eta(x)]=-\frac{1}{4\sqrt{x}}\ln\frac{\bar{S}(x)}{\bar{S}^*(x)}\equiv\frac{1}{i\sqrt{x}}\varphi(x), \quad (10)$$

where $\delta_I(x)$ is the imaginary part of the phase shift.

After substituting Eqs. (9) and (10) into Eq. (7) and noting that

$$\frac{1}{\pi}\int_0^\infty\frac{\ln|D(x)|}{2(x-z)\sqrt{x}}dx =-\frac{1}{2i\sqrt{z}}\ln D(z)+\frac{1}{2i\sqrt{z}} \times\ln\left\{\prod_p\frac{ia_p^{1/2}-z^{1/2}}{ia_p^{1/2}+z^{1/2}}\prod_s\frac{\alpha_s^{1/2}-z^{1/2}}{\alpha_s^{1/2}+z^{1/2}}\frac{(\alpha_s^*)^{1/2}-z^{1/2}}{(\alpha_s^*)^{1/2}+z^{1/2}}\right\}, \quad (11)$$

we finally obtain

$$\frac{\delta(z)}{\sqrt{z}}=\frac{1}{2i\sqrt{z}}\ln\left\{\prod_p\frac{ia_p^{1/2}-z^{1/2}}{ia_p^{1/2}+z^{1/2}}\prod_s\frac{\alpha_s^{1/2}-z^{1/2}}{\alpha_s^{1/2}+z^{1/2}}\frac{(\alpha_s^*)^{1/2}-z^{1/2}}{(\alpha_s^*)^{1/2}+z^{1/2}}\right\} +\frac{1}{\pi}\int_3^\infty\frac{\delta_I(x)dx}{(x-z)\sqrt{x}}+\frac{1}{\pi}\int_1^\infty\frac{\varphi(-x)dx}{(x+z)\sqrt{x}}, \quad (12)$$

where we have used the fact that $\delta_I(x)=0$ for $0\leqslant x\leqslant 3$ ($x=3$ being the lowest inelastic threshold in the case of pion kinematics).

Equation (12) provides the basis for all the considerations to follow. It has been obtained previously by various authors[5] and we present our derivation of it mainly for the sake of completeness and clarity. We remark that the structure of Eq. (12) is quite general and only the details depend on the assumption that we are dealing with the scattering of pions. The terms due to the zeros of $S(z)$ and the integral over the inelastic cut are essentially unchanged when one considers other systems; however, additional unphysical and/or kinematic cuts may appear. If bound states are present, the S matrix has poles in the gap; by suitably redefining $D(z)$ we can still construct an $\bar{S}(z)$ which has these poles eliminated and the derivation proceeds as before. Thus if there were a bound state in the pion-pion system at $z=-B$, $0<B<1$, there would appear on the right side of Eq. (12) the term $(2i\sqrt{z})^{-1}\ln[(i\sqrt{B}+\sqrt{z})/(i\sqrt{B}-\sqrt{z})]$.

Our $\delta(z)$ is the same as the physical scattering phase shift only in the elastic region: $z=k^2$, $0\leqslant k^2\leqslant 3$. Equation (12) is valid for $z=k^2$ provided that none of the zeros of the S matrix lie in the elastic interval. Since in the elastic region unitarity and reality imply that the magnitude of the S matrix is equal to unity, it is clear that the S matrix cannot vanish there. Setting $z=k^2$ and differentiating with respect to the momentum k we obtain from Eq. (12) the following expression for the momentum derivative of the phase shift:

$$\frac{d\delta(k^2)}{dk}=\sum_p\frac{(a_p)^{1/2}}{k^2+a_p}+\sum_s 2\,\mathrm{Im}(\alpha_s)^{1/2}\frac{k^2+|\alpha_s|}{|k^2-\alpha_s|^2} +\frac{1}{\pi}\int_3^\infty\frac{(x+k^2)\delta_I(x)}{(x-k^2)^2\sqrt{x}}dx +\frac{1}{\pi}\int_1^\infty\frac{(x-k^2)\varphi(-x)}{(x+k^2)^2\sqrt{x}}dx. \quad (13)$$

It follows from unitarity that $\delta_I(x)\geqslant 0$, $x\geqslant 0$, and consequently the contributions of all terms in Eq. (13) are non-negative, except for the integral over $\varphi(-x)$ which will be referred to in what follows as $I(k^2)$. Our next task, therefore, is to obtain a lower bound for $I(k^2)$.

3. BOUND ON THE MOMENTUM DERIVATIVE OF THE PHASE SHIFT

It follows from its definition, Eq. (10), that $2\varphi(x)$ is equal to the phase of $\bar{S}(x)$, up to an additive constant. Consequently, $2\varphi(x)$ changes at most by π as x varies between two consecutive zeros of odd order of $\mathrm{Im}\bar{S}(x)$, or of $\mathrm{Re}\bar{S}(x)$. Let such zeros of $\mathrm{Re}\bar{S}(x)$ that lie along the unphysical cut be denoted by $-\beta_j$, $j=1,2\cdots n$, labeled in order of increasing magnitude:

$$\beta_0<\beta_1<\cdots<\beta_n<\beta_{n+1}, \quad (14)$$

where we have set for convenience $1=\beta_0$, $\infty=\beta_{n+1}$. Let $\varphi_j^{\min(\max)}$ stand for the minimum (maximum) value of $\varphi(-x)$ for $\beta_j\leqslant x\leqslant\beta_{j+1}$. In order not to interrupt the argument we postpone the proof that

$$2\varphi_0^{\min}=-\pi/2 \quad (15)$$

from which it follows that

$$2\varphi_j^{\min}=-\pi(j+\tfrac{1}{2}),\quad 2\varphi_j^{\max}=+\pi(j+\tfrac{1}{2}). \quad (16)$$

We may now obtain a lower bound on $I(k^2)$ by

[5] J. C. Ball and W. R. Frazer, Phys. Rev. Letters 7, 204 (1961); and in particular T. Ogimoto, Progr. Theoret. Phys. (Kyoto) 27, 396 (1962); and C. H. Albright and W. D. McGlinn, Nuovo Cimento 25, 193 (1962).

breaking up the range of integration into intervals whose end points are the β_j and replacing $\varphi(-x)$ within each interval by $\varphi_j^{\min}$ (if $x>k^2$) or $\varphi_j^{\max}$ (if $x<k^2$). Thus, for $k^2\leqslant 1$ we have

$$I(k^2)>\sum_{j=0}^{n}\frac{\varphi_j^{\min}}{\pi}\int_{\beta_j}^{\beta_{j+1}}\frac{(x-k^2)dx}{(x+k^2)^2\sqrt{x}}$$
$$=\sum_{j=0}^{n}(j+\tfrac{1}{2})\left(\frac{(\beta_{j+1})^{1/2}}{\beta_{j+1}+k^2}-\frac{(\beta_j)^{1/2}}{\beta_j+k^2}\right)$$
$$=-\tfrac{1}{2}(1+k^2)^{-1}-\sum_{j=1}^{n}(\beta_j+k^2)^{-1}(\beta_j)^{1/2}. \quad (17)$$

Proceeding in a corresponding fashion, for $k^2>1$ we obtain

$$I(k^2)>\tfrac{1}{2}(1+k^2)^{-1}-\tfrac{1}{2}(1+2m)(|k|)^{-1}$$
$$-\sum_{j=1}^{n}(\beta_j+k^2)^{-1}(\beta_j)^{1/2}+2\sum_{j=1}^{m}(\beta_j+k^2)^{-1}(\beta_j)^{1/2}, \quad (18)$$

where m is the number of zeros that lie between 1 and k^2.

If only the numbers n and m, but not the positions $-\beta_j$, of the zeros are known, Eqs. (17) and (18) must be replaced by the weaker inequalities

$$I(k^2)>-(n+\tfrac{1}{2})(1+k^2)^{-1},\quad k^2\leqslant 1$$
$$>-\tfrac{1}{2}(n+m+1)(|k|)^{-1}+(m+\tfrac{1}{2})(1+k^2)^{-1},\quad k^2>1. \quad (19)$$

If desired, the dependence on m and k^2 may be eliminated from Eq. (19) by making use of the inequalities

$$0\leqslant m\leqslant n, \quad (20)$$

$$0\leqslant k^2\leqslant 3. \quad (21)$$

In this manner we obtain the rigorous lower bound on the momentum derivative of the phase shift as

$$d\delta/dk>-(n+\tfrac{1}{2}),\quad 0\leqslant k^2\leqslant 3, \quad (22)$$

where n is the number of zeros of the real part of the S matrix along the unphysical cut that are not simultaneously zeros of the imaginary part of S (this last restriction can, of course, be dropped since $n\leqslant N$ where N is the number of zeros of ReS without restrictions).

This completes our derivation of a bound on the momentum derivative of the phase shift except for the proof of Eq. (15). By definition, Eq. (10), $2\varphi(x)=0$, for x in the gap between 0 and -1. Hence $2\varphi(x+i0)$ for x just to the left of -1 is equal to the change in the phase of $\bar{S}(x+i0)$ as x moves from just to the left of -1 to the right of -1. Now it can be shown[6] that the partial-wave amplitude $f(z)$ has at $z=-1$ a square-root type of singularity with Im$f(z)$ vanishing like $(z+1)^{3/2}$. Consequently, in the neighborhood of $z=-1$ we may write

$$f(z)=(z+1)^{3/2}g(z)+h(z), \quad (23)$$

where $g(z)$ and $h(z)$ are real analytic functions. Therefore,

$$\bar{S}(z)\underset{z\to-1}{\longrightarrow}[1-2h(-1)(z+1)^{-1/2}]D^{-1}(-1). \quad (24)$$

In writing Eq. (24) we have used the fact that the zeros of $D(z)$ are the same as the zeros of $S(z)$ and so $D(-1)\neq 0$ since

$$S(-1)=\lim_{z\to-1}[1-2h(-1)(z+1)^{-1/2}]\neq 0. \quad (25)$$

Thus, if $h(-1)\neq 0$, the change in the phase of $\bar{S}(x+i0)$ as x goes from $-1-\epsilon$ to $-1+\epsilon$ is the same as the change in the phase of $(x+1+i0)^{-1/2}$, i.e., the phase increases from $-\pi/2$ to zero. If $h(-1)=0$, then there is no change in the phase of $\bar{S}(x+i0)$. In either case we have

$$2\varphi_0^{\min}=-\pi/2.$$

4. BOUND ON THE PION-PION COUPLING CONSTANT

The pion-pion coupling constant λ may be defined by[7]

$$\lambda=\textstyle\sum_l\lambda_l,$$
$$\lambda_l=-\tfrac{1}{5}(2l+1)P_l(0)f_l(-\tfrac{2}{3}), \quad (26)$$

where P_l is the lth Legendre polynomial and f_l is the lth partial-wave amplitude for the scattering of pions in the isotopic spin 0 state (the same equation holds for the isotopic spin 2 amplitudes provided that the factor $\frac{1}{5}$ is replaced by $\frac{1}{2}$).

It follows from Eq. (12) that

$$e^{2i\delta(-2/3)}=e^J\prod_p\frac{(a_p)^{1/2}-\gamma}{(a_p)^{1/2}+\gamma}\prod_s\left|\frac{(\alpha_s)^{1/2}-i\gamma}{(\alpha_s)^{1/2}+i\gamma}\right|^2$$
$$\times\exp\left\{-\frac{2\gamma}{\pi}\int_3^\infty\frac{\delta_I(x)dx}{(x+\frac{2}{3})\sqrt{x}}\right\}, \quad (27)$$

where $\gamma\equiv(\frac{2}{3})^{1/2}$ and

$$J\equiv-\frac{2\gamma}{\pi}\int_1^\infty\frac{\varphi(-x)dx}{(x-\frac{2}{3})\sqrt{x}}$$
$$<-\gamma\sum_{j=0}^{n}\frac{2\varphi_j^{\min}}{\pi}\int_{\beta_j}^{\beta_{j+1}}\frac{dx}{(x-\frac{2}{3})\sqrt{x}}$$
$$=\ln\left\{\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2}\prod_{j=1}^{n}\frac{(\beta_j)^{1/2}+\gamma}{(\beta_j)^{1/2}-\gamma}\right\}. \quad (28)$$

Hence

$$|e^{2i\delta(-2/3)}|<\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2}\prod_{j=1}^{n}\frac{(\beta_j)^{1/2}+\gamma}{(\beta_j)^{1/2}-\gamma}<\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2+n}, \quad (29)$$

where we must use the second form of the inequality

[6] R. Blankenbecler, M. L. Goldberger, S. W. McDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

(29) if only the number n, but not the positions $-\beta_j$, of the zeros is known.

Combining Eqs. (1), (26), and (29) we obtain bounds on λ_l. Thus, for example, we have for s waves

$$|10\sqrt{2}\lambda_0+1|<\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2+n_0}. \quad (30)$$

In particular, if $n_0=0$ we obtain

$$-0.29\lesssim\lambda_0\lesssim0.15, \quad (31)$$

which may be compared with the limits obtained by Chew and Mandelstam,[7] their Eq. (V.19):

$$-0.36\lesssim\lambda_0\lesssim0.3.$$

Since only even values of l contribute to Eq. (26), and since $f_l(x)$ vanishes like $x^{l+1/2}$ as $x\to0$, it is plausible to expect that all λ_l, $l\neq0$, can be neglected in comparison with λ_0. In that case, bounds on λ may be obtained from a consideration of s waves alone by setting $\lambda\approx\lambda_0$.

5. BOUND ON THE PION-NUCLEON COUPLING CONSTANT

In this section we shall use the same techniques as in previous sections to obtain an upper bound on the pion-nucleon coupling constant.

The analytic properties of the pion-nucleon partial-wave scattering amplitudes have been studied by Frazer and Fulco, and Frautschi and Walecka,[8] on the basis of the Mandelstam representation for the invariant amplitudes. They showed that the partial-wave amplitudes are analytic in the cut W plane, where W is the total energy in the barycentric frame, with the cuts extending into the complex plane as well as onto the real axis. It is easily seen, however, that the partial-wave amplitudes are analytic in the cut ω plane, where ω is the pion energy in the barycentric frame, with the cuts lying only on the real and imaginary axes. This analytic behavior is similar to the one that can be derived from the fixed source Chew-Low model,[9] in which the singularities along the imaginary axis are due to the cutoff function instead of being due to the crossed process $\pi\pi\to N\bar{N}$ of the relativistic dispersion relations.

As will be seen later, the imaginary cut of the pion-nucleon scattering plays a similar role, as far as the dispersion relation for the phase shift is concerned, as the left-hand cut of the previous sections. Since the contributions from these cuts are not known *a priori*, there is little difference between the Chew-Low model and the relativistic dispersion model. For this reason we confine ourselves to the Chew-Low model which is easier to handle.

The scattering amplitude, h_α, has the following representation:

$$h_\alpha(\omega)=\frac{\lambda_\alpha}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega_p\left[\frac{\mathrm{Im}h_\alpha(\omega_p)}{\omega_p-\omega}+\sum_\beta A_{\alpha\beta}\frac{\mathrm{Im}h_\beta(\omega_p)}{\omega_p+\omega}\right], \quad (32)$$

where p is the pion momentum and $\alpha=1,2,3$ corresponds to the states $T=\frac{1}{2}\,P_{1/2}$, $T=\frac{1}{2}\,P_{3/2}$ or $T=\frac{3}{2}\,P_{1/2}$, and $T=\frac{3}{2}\,P_{3/2}$, respectively. The cutoff function $v(\omega)$ is assumed to have the representation

$$v(\omega)=\int_{m_0^2}^\infty dm^2\,\sigma(m^2)/(m^2+\omega^2),$$
$$\int_{m_0^2}^\infty dm^2\,\sigma(m^2)/m^2=1. \quad (33)$$

The Born term λ_α is given by

$$\lambda_\alpha=\tfrac{2}{3}f^2\begin{bmatrix}-4\\-1\\2\end{bmatrix}, \quad (34)$$

where f is the pion-nucleon coupling constant. The h_α may be expressed in terms of the phase shifts δ_α:

$$\lim_{\omega\to\omega_p+i\epsilon} h_\alpha(\omega)=e^{i\delta_\alpha(p)}\sin\delta_\alpha(p)/[p^3v^2(\omega_p)], \quad (35)$$

and satisfy the crossing relation:

$$h_\alpha(-\omega)=\textstyle\sum_\beta A_{\alpha\beta}h_\beta(\omega), \quad (36)$$

where

$$A=\tfrac{1}{9}\begin{bmatrix}1&-8&16\\-2&7&4\\4&4&1\end{bmatrix}. \quad (37)$$

Let us look at the analytic structure of $\eta_\alpha(\omega)$, the scattering phase shift divided by the momentum:

$$\eta_\alpha(\omega)=(1/2ip)\ln S_\alpha(\omega),$$
$$S_\alpha(\omega)=1+2ip^3v^2(\omega)h_\alpha(\omega). \quad (38)$$

It follows from Eqs. (32), (33), and (38) that $\eta_\alpha(\omega)$ has a logarithmic branch point at $\omega=0$, a cut along the positive real axis for $\omega\geq2$ and a cut along the negative real axis for $\omega\leqslant-1$, and a pair of cuts along the imaginary axis due to the cutoff function. In addition to these singularities η_α has logarithmic branch points at the zeros of S_α. Since $S_\alpha(\pm1)=1$ and $S_\alpha(0\pm\epsilon)=\pm\infty$ for $\alpha=3$ ($\mp\infty$ for $\alpha=1,2$), S_α must have at least one zero between -1 and 0 for $\alpha=3$ (between 0 and 1 for $\alpha=1,2$) because by the representation, Eq. (32), S_α is a real continuous function of ω in both the regions $(-1, 0-)$ and $(0+, 1)$. We note that S_3 cannot have zeros between 0 and 1 because h_3 is positive definite in that region. In addition to the real zeros, S_α may, in general, have complex zeros which occur in conjugate pairs. Let

[8] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960); S. C. Frautschi and J. D. Walecka, *ibid.* **120**, 1486 (1960).

[9] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

us assume for simplicity that S_α has just one real zero between -1 and $+1$ at $\omega=-\kappa_\alpha$, and just one pair of complex zeros at $\omega=\omega_r$ and $\omega=\omega_r^*$.[10]

By proceeding in the same manner as in previous sections the following dispersion relation for η_α can be derived:

$$\eta_\alpha(\omega)=\frac{1}{2ip}\ln(S_BS_R)+\frac{1}{\pi}\int_1^\infty\frac{\mathrm{Im}\chi_\alpha(\omega')}{\omega'+\omega}d\omega' +\frac{1}{\pi}\int_2^\infty\frac{\mathrm{Im}\eta_\alpha(\omega')}{\omega'-\omega}d\omega'+\frac{1}{\pi}\int_{m_0}^\infty\frac{dm}{m^2+\omega^2} \times[m\,\mathrm{Im}\tau_\alpha(m)-\omega\,\mathrm{Re}\tau_\alpha(m)], \quad (39)$$

where S_B is the portion of the S matrix due to the Born term and the real zero:

$$S_B=\frac{\omega\zeta+\omega+\kappa-i\kappa p}{\omega\zeta+\omega+\kappa+i\kappa p},\quad \zeta\equiv(1-\kappa^2)^{1/2}, \quad (40)$$

and S_R is the portion of the S matrix due to the pair of complex zeros:

$$S_R=\frac{(k_r+k_r^*)(\omega-\omega_r)-(\omega_r-\omega_r^*)(p-k_r)}{(k_r+k_r^*)(\omega-\omega_r^*)-(\omega_r^*-\omega_r)(p-k_r^*)}, \quad (41)$$

where $k_r^2+1=\omega_r^2$ and $\mathrm{Im}k_r>0$.

The second term in Eq. (39) is due to the left-hand cut with $\chi_\alpha(\omega)$ given by

$$\chi_\alpha(\omega)=\eta_\alpha(-\omega). \quad (42)$$

Using Eqs. (35), (36), and (38) we obtain

$$\chi_\alpha(\omega)=(1/2ip)\ln(\textstyle\sum_\beta A_{\alpha\beta}e^{2i\delta_\beta(\omega)}), \quad (43)$$

so that

$$\mathrm{Im}\chi_\alpha(\omega)=-\frac{1}{2p}\ln|\sum_\beta A_{\alpha\beta}e^{2i\delta_\beta(\omega)}|. \quad (44)$$

For $\alpha=3$ we obtain

$$\mathrm{Im}\chi_\alpha(\omega)\geqslant 0, \quad (45)$$

by making use of Eq. (44) and the inequality

$$|\textstyle\sum_\beta A_{3\beta}e^{2i\delta_\beta(\omega)}|\leqslant\sum_\beta|A_{3\beta}|=1. \quad (46)$$

The third term in Eq. (39) is due to the right-hand cut. It starts at the first inelastic threshold $\omega=2$ because $\mathrm{Im}\eta_\alpha(\omega)=0$ for $1\leqslant\omega\leqslant 2$. We also note that by unitarity

$$\mathrm{Im}\eta_\alpha(\omega)\geqslant 0. \quad (47)$$

The last term in Eq. (39) is due to the cuts along the imaginary axis. It is determined entirely by the cutoff function with τ given by

$$\tau_\alpha(m)=\frac{1}{2(m^2+1)^{1/2}} \times\ln\frac{1+2(m^2+1)^{3/2}h_\alpha(im)v^2(im+\epsilon)}{1+2(m^2+1)^{3/2}h_\alpha(im)v^2(im-\epsilon)}. \quad (48)$$

To obtain an upper bound on the magnitude of the coupling constant we make use of the dispersion relation for η_α in the (3,3) state in order to take advantage of the inequality (45). It then follows from Eq. (39) and from

$$\lim_{\omega\to 0}\left[\eta_3(\omega)+\frac{1}{2ip}\ln\omega\right]=-\frac{1}{2}\ln\left(\frac{8}{3}f^2\right) \quad (49)$$

that

$$f^2\leqslant\tfrac{3}{4}[\kappa/(1+\zeta)]\alpha A, \quad (50)$$

where

$$\alpha=\frac{\omega_r^*k_r+\omega_rk_r^*+i(\omega_r-\omega_r^*)}{\omega_r^*k_r+\omega_rk_r^*-i(\omega_r-\omega_r^*)},$$
$$A=\exp\left\{-(2/\pi)\int_{m_0}^\infty dm\,\mathrm{Im}\tau(m)/m\right\}. \quad (51)$$

It follows from Eq. (38) that $\eta_\alpha(\pm1)=0$. Imposing this requirement on Eq. (39) and using Eqs. (45) and (47) we obtain the inequality

$$\kappa^2/[2\zeta(1+\zeta)]+\beta-B\leqslant 0, \quad (52)$$

where

$$\beta=\frac{\omega_r-\omega_r^*}{i}\frac{\omega_r^*k_r+\omega_rk_r^*}{(\omega_r^*k_r+\omega_rk_r^*)^2-(k_r+k_r^*)^2},$$
$$B=-(1/\pi)\int_{m_0}^\infty dm\,m\,\mathrm{Im}\tau(m)/(m^2+1). \quad (53)$$

Using Eq. (52) to eliminate $\kappa/(1+\zeta)$ from Eq. (50) we obtain

$$f^2\leqslant\frac{3}{4}\left(\frac{B-\beta}{1+B-\beta}\right)^{1/2}\alpha A.$$

It now remains to obtain bounds on A and B. This requires a knowledge of $\mathrm{Im}\tau$, which plays here the same role as φ did in the $\pi\pi$ scattering problem. The estimate of upper and lower bounds on $\mathrm{Im}\tau$, although straightforward, is more complicated than the corresponding estimate for φ. In the present case we must know the number of zeros, n, of the following quantity (which corresponds to $\mathrm{Im}\bar{S}$ of Sec. 3):

$$(\mathrm{Re}v)(\mathrm{Im}v)\{\mathrm{Re}h(im)+2(m^2+1)^{3/2}|h(im)|^2 \times[(\mathrm{Re}v)^2-(\mathrm{Im}v)^2]\}, \quad (55)$$

where the argument of v is $im+\epsilon$.

[10] Additional zeros may of course be included. Their only effect is to strengthen the inequalities (50) and (52).

The imaginary part of τ is then bounded by

$$|\mathrm{Im}\tau|\leqslant\frac{(n+1)\pi}{2(m^2+1)^{1/2}}, \tag{56}$$

and A and B are bounded by

$$A\leqslant\left(\frac{m_0+(m_0^2+1)^{1/2}+1}{m_0+(m_0^2+1)^{1/2}-1}\right)^{n+1},$$
$$B\leqslant\tfrac{1}{2}(n+1)(m_0^2+1)^{-1/2}. \tag{57}$$

Thus, we finally obtain

$$f^2\leqslant\tfrac{3}{4}\alpha\left(\frac{n+1-2\beta(m_0^2+1)^{1/2}}{2(m_0^2+1)^{1/2}(1-\beta)+n+1}\right)^{1/2}\times\left(\frac{m_0+(m_0^2+1)^{1/2}+1}{m_0+(m_0^2+1)^{1/2}-1}\right)^{n+1}. \tag{58}$$

For a Feynman-type cutoff, i.e.,

$$\sigma(m^2)=m_0^2\delta(m^2-m_0^2), \tag{59}$$

we find $n=3$ and therefore $f^2\leqslant 0.36$, when α and β are calculated from the (3,3) resonance data and m_0 is chosen to be the nucleon mass.

6. DISCUSSION

The main purpose of this work was to show that there exists in field theory an analog to Wigner's theorem, in spite of the fact that the concept of a finite interaction range is not meaningful. It might be well to review here the assumptions that were made to obtain this result.

First, the validity of the Mandelstam representation was assumed thus providing us with the information about the analytic structure of the S matrix. Clearly the analytic structure of the S matrix could be considerably more complicated without invalidating our results. Thus the use of the Mandelstam representation should not be regarded as being essential but rather as supplying a convenient framework.

Second, it was assumed that the function $\bar{S}(z)$, defined by Eq. (5), exists, i.e., that the product $D(z)$ has meaning. This product is obviously meaningful if the number of zeros of the S matrix is finite. The situation is more involved if the S matrix has an infinite number of zeros. According to the Weierstrass factor theorem,[11] an entire function $G(z)$ with an infinite number of zeros, none of which are at $z=0$, may be represented as

$$G(z)=\prod_{\nu=1}^{\infty}\left\{(1-z/z_\nu)\exp\left[(z/z_\nu)+\tfrac{1}{2}(z/z_\nu)^2+\cdots+\frac{1}{k_\nu-1}(z/z_\nu)^{k_\nu-1}\right]\right\}, \tag{60}$$

with the integers k_ν chosen in such a way that

$$\sum_{\nu=1}^{\infty}(z/z_\nu)^{k_\nu} \tag{61}$$

is absolutely convergent for every z. Here the zeros of $G(z)$ occur at z_ν and the z_ν for different ν are not necessarily distinct. Since the use of $D(z)$ in Eq. (5) corresponds to setting all k_ν in Eq. (60) equal to unity, it follows that we have assumed that if the number of zeros of the S matrix is infinite then the spacing between them increases sufficiently rapidly so that

$$\sum_{\nu=1}^{\infty}|z/z_\nu|<\infty. \tag{62}$$

Although this restriction can be relaxed the resultant bounds on the momentum derivative of the phase shift become quite involved and so we prefer to keep Eq. (62) as one of the conditions needed to obtain our results.

Third, we have assumed that $\delta(z)$ behaves at infinity no worse than $z^{1/2-\epsilon}$, $\epsilon>0$. This actually is not an additional assumption but follows from the previous two, since if $D(z)$ is finite and the Mandelstam representation is valid then $\delta(z)$ can diverge no worse than logarithmically.

Lastly, our result as expressed in terms of n [for example Eq. (22)] will be useful only if n is finite, n being the number of zeros of, for example, $\mathrm{Re}\bar{S}(z)$ along the unphysical cut. Should n be infinite we can still obtain a bound by using the inequalities that involve explicitly the positions of these zeros [for example, Eqs. (17) and (18)], provided that the infinite sums or products over the number of these zeros converge.

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[11] K. Knopp, *Theory of Functions* (Dover Publications, New York, 1947), Part II.